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ON A CLASS OF LATTICES ASSOCIATED WITH n-CUBES*

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A Lattice $L(X)$ is defined starting from a cubical lattice L and an increasing diagonally closed subset X of L (Section 1). The lattice $L(X)$ are proved to be—up to isomorphism—precisely those of signed simplexes of a simplical complex (Section 2); furthermore, an algebraic combinatorial characterization of the lattices $L(X)$ is given (Section 3).

1. Introduction

Notation and terminology in this paper are the usual lattice-theoretic ones; in particular, $\sup L$, and $\inf L$ will always denoted by I and 0, respectively.

Let L_n be an n-dimensional cubical lattice, i.e. the lattice of the faces of an n-cube. Recall that any proper interval $[x, \mathrm{I}]$ in L_n is isomorphic to a Boolean algebra, whereas any interval $[0, x]$ is isomorphic to a cubical lattice L_m, with $m \leq n$. In what follows L_n will be taken as the lattice of the faces in an n-cube defined by the fundamental points of a reference frame in an n-dimensional Euclidean space E_n; $x \leq y$ will mean that the face x of L_n is contained in the face y.

For any x in L_n, *the diagonal with respect to* x is the mapping Δ_x of the interval $[0, x]$ onto itself which maps 0 onto 0 and any non-empty face $y \in [0, x]$ onto the unique face opposite to y in the cube determined by $[0, x]$. The diagonals Δ_x with respect to the elements x in L are involutorial automorphisms, i.e. $\Delta_x\Delta_x(y) = y$, of the intervals $[0, x]$ and satisfy the following properties:

(i) if $a < x$, then $a \wedge \Delta_x(a) = 0$;

(i)′ if $a < x$ and $b < x$, then the two conditions

$$\Delta_x(a) \vee b < x \quad \text{and} \quad a \wedge b = 0$$

are equivalent.

In [4] the following result is proved.

Theorem (Metropolis–Rota). *Let L be a finite lattice with* $\sup L = \mathrm{I}$ *and* $\inf L = 0$ *and for any x in L let Δ_x be an involutorial automorphism of the interval* $[0, x]$.

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Then, up to isomorphism, L is the cubical lattice L_n, for some integer n, and $\{\Delta_x\}$ is the family of the diagonals of L_n, iff (i) *and* (i)′ *hold.*

A non-empty subset X of L_n is said to be *diagonally closed* if for any x, y in L_n, $x \leq y$, the element $\Delta_y(x)$ belongs to X. If X is an increasing diagonally closed subset, the partial order on L_n, induces a lattice structure on $X \cup \{0\}$. This lattice will be denoted by $L(X)$ and said to be *associated with* X. Recall that X is an increasing subset iff $x \in X$ and $x \leq y$ implies $y \in X$. If $x \in L(X)$, the restriction of Δ_x to the interval defined by 0 and x in $L(X)$ is an automorphism of this interval which will be called the *diagonal with respect to x in* $L(X)$. Such an automorphism satisfies (i) but in the general case it does not satisfy (i)′ since $L(X)$ cannot be a cubical lattice.

In what follows $D(L_n)$ will denote the family of increasing diagonally closed subsets of L_n.

This paper was inspired by [3] and [4], whose notation will be partly used, and its aim is the investigation of the type $L(X)$ lattices, with $X \in D(L_n)$. Thus, in the following section it will be proved that, up to isomorphism, these lattices are precisely the lattices of signed simplexes of a simplicial complex and in Section 3 an algebraic combinatorial characterization of them will be given.

2. A representation of the lattices $L(X)$ with $X \in D(L_n)$

Let S be a finite set of order n, which for convenience sake will always be identified with $\{1, 2, \ldots, n\}$, and let K be a simplicial complex whose vertex set is a non-empty subset of S. A *signed simplex* of K is an ordered pair $A_\sigma = (A_1, A_2)$ of disjoint subsets of S such that $A_1 \cup A_2$ is a simplex of K. The set $L'(K)$ consisting of all signed simplexes in K can be ordered by the following relation $\leq$

$$A_\sigma = (A_1, A_2) \leq (B_1, B_2) = B_\sigma \quad \Leftrightarrow \quad A_1 \supseteq B_1 \text{ and } A_2 \supseteq B_2.$$

If to $L'(K)$ a least element 0 is added, then a lattice $L(K)$ is obtained which will be called the *lattice of signed simplexes in* K. The binary operations $\vee$ (join) and $\wedge$ (meet) on $L(K)$ are defined as follows for the elements other than 0:

$$A_\sigma \vee B_\sigma = (A_1 \cap B_1, A_2 \cap B_2);$$

$$A_\sigma \wedge B_\sigma = \begin{cases} (A_1 \cup B_1, A_2 \cup B_2) & \text{if } (A_1 \cup B_1) \cap (A_2 \cup B_2) = \emptyset \text{ and} \\ & \quad A_1 \cup A_2 \cup B_1 \cup B_2 \text{ is a simplex of } K, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $\sup L(K) = \mathrm{I}$ is the signed simplex $(\emptyset, \emptyset)$, where $\emptyset$ denotes the empty face of K. For any element A in $L(K)$, the *diagonal with respect to* A is defined to be the mapping $\Delta(A_\sigma, \cdot)$ of the interval $[0, A_\sigma]$ onto itself such that for any B_σ in

$[0, A_\sigma]$,

$$\Delta(A_\sigma, B_\sigma) = \begin{cases} (A_1 \cup (B_2 - A_2), A_2 \cup (B_1 - A_1)) & \text{if } B_\sigma \neq 0, \\ 0 & \text{if } B_\sigma = 0. \end{cases}$$

It is easy to check that $\Delta(A_\sigma, \cdot)$ is an involutorial automorphism of the interval $[0, A_\sigma]$ for which (i) holds.

Next, let Ω'_K be the mapping of $L(K)-\{0\}$ into L_n defined as follows. For any signed simplex $A_\sigma = (A_1, A_2) \neq 0$, I, $\Omega'_K(A_\sigma)$ is the face of L_n at which the hyperplanes $x_i = 1$, $x_j = 0$, $(i, j) \in A_1 \times A_2$, in E_n meet; furthermore, $\Omega'_K(\mathrm{I}) = \mathrm{I}$. It is straightforward to check that Ω'_K is an order preserving monomorphism between $L(K)-\{0\}$ and L_n and that $X = \Omega'_K(L(K)-\{0\})$ is an increasing diagonally closed subset of L_n. Moreover, the mapping $\Omega_K : L(K) \to L(X)$, which maps zero onto zero and acts as Ω'_K on the other elements of $L(K)$ is a lattice isomorphism between $L(K)$ and $L(X)$. Finally, if—abusing notation—we denote by Δ_x the diagonal in $L(X)$ with respect to an element $x \in L(X)$ (Section 1), then

$$\Omega_K(\Delta(A_\sigma, B_\sigma)) = \Delta_{\Omega_K(A_\sigma)}(\Omega_K(B_\sigma));$$

therefore, Ω_K preserves diagonals.

Conversely, for $X \in D(L_n)$, let K be the family of subsets of S consisting of the empty set and the subsets A defined as follows:

$A \neq \emptyset, A \in K \Leftrightarrow$ A can be partitioned into two disjoint blocks $\{i_1, i_2, \ldots, i_h\}$, $\{j_1, j_2, \ldots, j_k\}$ such that the face of L_n with equations $\{x_{i_1} = x_{i_2} = \cdots = x_{i_h} = 1, x_{j_1} = x_{j_2} = \cdots = x_{j_k} = 0\}$ belongs to $L(X)$.

The above defined family K turns out to be a simplicial complex and $\Omega'_K(L(K)-\{0\}) = X$. Furthermore, it is straightforward that, for any simplicial complex K, the simplicial complex associated with $\Omega'_K(L(K)-\{0\})$ is precisely K. Consequently, the following result is proved.

Theorem 1. *The lattices of signed simplexes of simplicial complexes together with their respective diagonals are, up to isomorphism, exactly the lattices associated with the increasing and diagonally closed subsets of cubical lattices together with their respective diagonals.*

In the special case K is the Boolean algebra of all subsets of S, Theorem 1 gives again the well known result that L_n is isomorphic to the lattice of signed subsets of a finite set of order n (See [3, 4]).

3. A characterization of lattices $L(X)$ with $X \in D(L_n)$

Let L be a finite co-atomic lattice, i.e.

(ii) *any element in L other than* I *is an inf of co-atoms,* and for any x in L let Δ_x be an involutorial automorphism of the interval $[0, x]$ for which (i) holds. A set C of co-atoms in L will be said to be of *cubical type* if

(γ_1) a subset A exists of C such that $\bigwedge\{a: a \in A\}$ is an atom and $C = A \cup \Delta_{\mathrm{I}}(A)$;

(γ_2) C is maximal with respect to property (γ_1).

Let C be a cubical type set of co-atoms and denote by $\bar{C}$ the lattice consisting of I and all inf's of the subsets of C. Moreover, let $\mathscr{C}(L)$ denote the family of all cubical type sets of co-atoms in L. Assume that the following property holds for L:

(iii) if $0 < a < x$ and $a = x \wedge c_1 \wedge c_2 \wedge \cdots \wedge c_h$, $c_1, c_2, \ldots, c_h$ being distinct co-atoms, then

$$\Delta_x(a) = x \wedge \Delta_{\mathrm{I}}(c_1) \wedge \cdots \wedge \Delta_{\mathrm{I}}(c_h).$$

Notice that, by (iii), the next result is true.

Proposition 1. *If $C \in \mathscr{C}(L)$ and $a, x \in \bar{C}$, $a \leq x$, then $\Delta_x(a)$ is an element of $\bar{C}$.*

Next, it will be shown that if (i) holds for a lattice, then (ii) and (iii) are independent. To prove this claim, consider the lattice L' in Fig. 1 and define the automorphism Δ_{I} as follows:

$$\Delta_{\mathrm{I}}(\mathrm{I}) = \mathrm{I}, \qquad \Delta_{\mathrm{I}}(0) = 0, \qquad \Delta_{\mathrm{I}}(c_i) = c_i',$$
$$\Delta_{\mathrm{I}}(c_i') = c_i, \qquad \Delta_{\mathrm{I}}(a_i) = a_i', \qquad \Delta_{\mathrm{I}}(a_i') = a_i.$$

Moreover, define the remaining involutorial automorphism in the unique possible way. For the lattice L' (ii) does not hold, whereas (iii) trivially holds.

Next, consider the lattice L'' in Fig. 2 and notice that (ii) holds. It is easy to check that (iii) cannot be satisfied if the automorphisms Δ_x are defined in a way that (i) holds. Thus, (ii) *and* (iii) *are independent.*

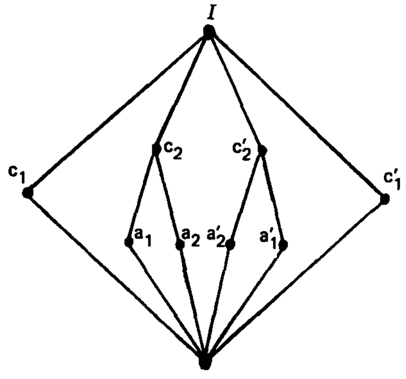

Fig. 1.

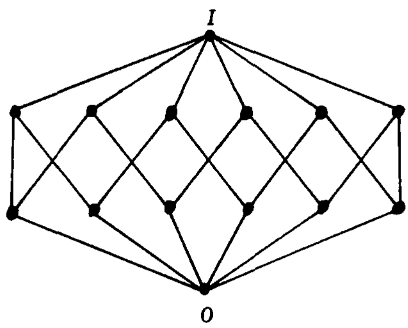

Fig. 2.

Finally, Theorem 2 provides the algebraic combinatorial characterization of the lattices $L(X)$, with $X \in D(L_n)$ mentioned in Section 1.

Theorem 2. *Let L be a co-atomic finite lattice with* $\sup L = \mathrm{I}$ *and* $\inf L = 0$ *and for any $x \in L$ let Δ_x be an involutorial automorphism of the interval $[0, x]$. Up to isomorphism, L is the lattice $L(X)$ associated with an increasing diagonally closed subset of L_n, for some integer n and some $X \in D(L_n)$, and $\{\Delta_x\}$ is the set of diagonals of $L(X)$ iff* (i) *and* (iii) *hold.*

Proof. The result is achieved through the following steps.

Step 1. *Each element x in L belongs to some lattice $\bar{C}$, where C is a set of cubical type co-atoms.*

The statement is obviously true when $x = 0$, I. Assume $x \neq 0$, I and let $a \leq x$ be an atom. Denote by A the set of all co-atoms greater than or equal to a; thus, $C = A \cup \Delta_{\mathrm{I}}(A)$ is of cubical type and, by (ii), $x \in \bar{C}$.

Step 2. *Any element $x \neq 0$, I is a unique irredundant meet of co-atoms.*

Let c_1, c_2, c_3 be distinct co-atoms and $x = c_1 \wedge c_2 \wedge c_3 = c_1 \wedge c_2 \neq 0$; thus,

$$y = \Delta_{\mathrm{I}}(x) = \Delta_{\mathrm{I}}(c_1) \wedge \Delta_{\mathrm{I}}(c_2) \wedge \Delta_{\mathrm{I}}(c_3) = \Delta_{\mathrm{I}}(c_1) \wedge \Delta_{\mathrm{I}}(c_2) \neq 0;$$

set $c_1' = \Delta_{\mathrm{I}}(c_1)$, $c_2' = \Delta_{\mathrm{I}}(c_2)$, $c_3' = \Delta_{\mathrm{I}}(c_3)$. The element $\Delta_{c_3'}(y) = c_3' \wedge c_1 \wedge c_2$ is less than or equal to $x = c_1 \wedge c_2$; on the other hand, $x = c_1 \wedge c_2 \wedge c_3$, therefore $\Delta_{c_3'}(y) \leq c_3$. Consequently, $\Delta_{c_3'}(y) = 0$ since $\Delta_{c_3'}(y) \leq c_3 \wedge c_3' = 0$, a contradiction as $\Delta_{c_3'}(y) \neq 0$.

Hence, if c_1, c_2, c_3 are co-atoms and $c_1 \wedge c_2 = c_1 \wedge c_3 \neq 0$, with $c_1 \neq c_2, c_3$, then $c_2 = c_3$; an induction argument proves the statement.

Step 3. *If $C = A \cup \Delta_I(A)$ is a set of cubical type co-atoms, then, for any subset B of C such that $B \cup \Delta_I(B) = C$, $\bigwedge \{b \colon b \in B\}$ is an atom of L.*

If $A = \{c_1, c_2, \ldots, c_t\}$, then $a = c_1 \wedge c_2 \wedge \cdots \wedge c_t$ is an atom of L, by definition; thus, for any $x > a$, $\Delta_x(a)$ is again an atom of L. Set $\Delta_{\mathrm{I}}(c_i) = c_i'$, $i = 1, 2, \ldots, t$. Assume $B = \{c_1, \ldots, c_h, c_{h+1}', \ldots, c_t'\}$, then the statement follows from the equality

$$\Delta_{c_1 \wedge c_2 \wedge \cdots \wedge c_h}(a) = c_1 \wedge c_2 \wedge \cdots \wedge c_h \wedge c_{h+1}' \wedge \cdots \wedge c_t'.$$

Step 4. Take any partition of the set C_L of all co-atoms in L into two disjoint blocks such as S and $\Delta_I(S)$. Let K be the simplicial complex whose maximal simplexes are the subsets M of S such that $M \cup \Delta_I(M)$ is of cubical type; moreover, let $L(K)$ be the lattice of signed simplexes in K.

Denote by Φ the mapping of $L(K)$ onto L which maps 0 and I onto 0 and I, respectively, and such that for any signed simplex $A_\sigma = (\{c_{i_1}, \ldots, c_{i_h}\}, \{c_{j_1}, \ldots, c_{j_k}\})$,

$$\Phi(A_\sigma) = c_{i_1} \wedge \cdots \wedge c_{i_h} \wedge \Delta_I(c_{j_1}) \wedge \cdots \wedge \Delta_I(c_{j_k}).$$

Taking into account (1), (2) and (3), it is easy to prove that Φ is a lattice isomorphism and

$$\Phi(\Delta(A_\sigma, B_\sigma)) = \Delta_{\Phi(A_\sigma)}(\Phi(B_\sigma)), \quad \text{for any } B_\sigma \leqslant A_\sigma.$$

Thus, by Theorem 1, the statement follows. □

Finally, notice that (i)′ (see Section 1) is equivalent to (ii), (iii) and C_L being of cubical type. Therefore, Theorem 2 can be viewed as a generalization of the Metropolis–Rota Theorem (see Section 1) and, under the assumption that the set C_L of all co-atoms in L is of cubical type, provides a new characterization of cubical lattices.

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